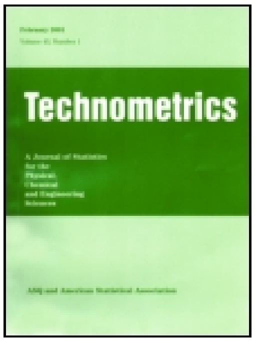

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Sequential Bayesian Design for Accelerated Life Tests

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Abstract

Most of the recently developed methods on optimum planning for accelerated life tests (ALT) involve "guessing" values of parameters to be estimated, and substituting such guesses in the proposed solution to obtain the final testing plan. In reality, such guesses may be very different from true values of the parameters, leading to inefficient test plans. To address this problem, we propose a sequential Bayesian strategy for planning of ALTs and a Bayesian estimation procedure for updating the parameter estimates sequentially. The proposed approach is motivated by ALT for polymer composite materials, but are generally applicable to a wide range of testing scenarios. Through the proposed sequential Bayesian design, one can efficiently collect data and then make predictions for the field performance. We use extensive simulations to evaluate the properties of the proposed sequential test planning strategy. We compare the proposed method to various traditional non-sequential optimum designs. Our results show that the proposed strategy is more robust and efficient, as compared to existing non-sequential optimum designs. The supplementary material for this paper is available online.

Key Words: Accelerated life test; Bayesian test planning; Fatigue testing; MCMC; Optimum design; Sequential design.

1 Introduction

1.1 Motivation

For the long-term durability of reliable products, it is critical to assess the reliability information such as the material's lifetime under specified stress levels. An accelerated life test (ALT) is often used to collect failure information in a timely manner by adopting stress levels that are higher than the normal stress level (i.e., the use condition). In a typical ALT setting, n experimental units are tested. Unit i is subjected to a specific level of stress x_i (the design input), and the failure time t_i (the response) is recorded. The purpose of an ALT is to make a precise prediction of a lifetime metric at the use condition, through a lifetime model connecting the failure time t to the accelerating variable x.

The problem in this field is, how to choose the input stress levels $x_1, \ldots, x_n$ for the n experimental units to estimate the desired characteristics (specifically, the pth quantile) associated with the failure distribution most efficiently. Whereas such a problem has been addressed in traditional ALT applications, it is particularly challenging for ALT with the following characteristics. First, even under elevated testing stress, the testing process can last several weeks, even months to observe a failure, making data collection a time-consuming and expensive process. Second, most testing laboratories are typically equipped with only one or two testing machines, making the task of testing multiple samples in parallel a near-impossible task. Third, while some prior information is available on the performance of new materials, such information typically involves considerable uncertainty due to lack of reference in literature. Typical examples of such ALT are polymer composites. In experimental design literature, optimal designs are typically obtained by maximizing a utility function based on the available "information" (e.g., Fisher information). However, in ALT experiments, the failure models are typically nonlinear, making the information dependent on the true value of the underlying parameters to be estimated. Therefore, the objective functions based on such information cannot be optimized without some prior knowledge about the true values of the parameters. There are two general classes of test planning methods: non-Bayesian and Bayesian. In traditional life test planning, the optimum designs are determined using a non-Bayesian approach, assuming that the true parameters are known (e.g., Meeker and Nelson, 1975; Meeker, 1984; Nelson, 1990, Ch. 6; Pascual, 2003; King et al., 2016). However, prior information about the true parameters is typically limited and involves considerable uncertainty. Moreover, due to the limited budget for testing, the number of samples that can be tested is often limited, so that combining information from different sources is a useful strategy. Thus, Bayesian techniques are more natural in life test planning. Such designs combine prior knowledge of parameters to design an efficient experiment, and then to make statistical inferences.

Further, the lack of parallel testing facilities, and expensive nature of tests make a sequential testing procedure almost inevitable. In a typical sequential procedure, the $(n+1)$th input point x_{n+1} is chosen by optimizing a utility function (information) computed from the data (x_i, t_i) for $i = 1, \ldots, n$. Sequential Bayesian designs, in which the subsequent design point is determined by optimizing an expected utility function over prior distributions of parameters, are intuitively appealing in our setting. In this paper, we propose a sequential Bayesian procedure to obtain an efficient fatigue testing plan for ALT with particular application to polymer composites.

1.2 State-of-the-art Designs for Polymer Composite ALT

A common application of ALT is the accelerated fatigue testing for polymer composite materials. Polymer composites are made either by combining different types of polymers or by combining polymers with other kinds of materials. They have many desirable properties, such as light weight, high strength, and long-term durability, and find applications in a wide array of industries including aircraft, wind turbine, transportation, construction, and even products used in daily life. However, the fatigue and other properties of the materials need to be tested, as they are required to meet certain industrial standards. Fatigue occurs when a material is exposed to varying levels of stress over a period of time. The majority of testing performed in this field is based on the standards provided in ASTM E739-10 (2010) for stress-based fatigue testing. More details about the test setup is available in Section 2.1.

According to the standard (ASTM E739-10, 2010), engineers usually use a balanced, equally-spaced design for ALT. This means, if there is a range of input stress, say $[x_L, x_U]$, to be applied, then a number of equally spaced points (usually four points) are chosen in this interval for the experiment, and each level of stress is applied to an equal number of experimental units.

Traditional non-Bayesian methods for designing efficient ALT are based on properties of maximum likelihood (ML) estimators. Meeker and Escobar (1998) provided a general guideline for planning life tests to obtain the precise prediction of the pth quantile at the use condition. Several authors such as Chernoff (1962) and Meeker and Hahn (1977; 1985) studied optimum and compromise ALT plans and outlined practical guidelines for planning an efficient ALT. Pascual (2003; 2004) considered optimal test plans for random fatigue-limit models. Recently, King et al. (2016) proposed optimum test planning techniques for polymer composites fatigue studies. Such optimal designs depend on the parameter values. Therefore, to identify the optimal input points, it is necessary to substitute some pre-assumed values of the parameters.

These designs work well if the parameter values substituted are reasonably close to the "true" ones, a condition that is difficult to guarantee. In Section 5, we will compare our proposed sequential Bayesian strategy to existing strategies, that can be considered benchmark strategies for planning ALT for polymer composite fatigue testing, and explain why the proposed strategy is more natural and robust in the type of setting we consider.

1.3 Other Related Literature

Here we provide a brief review of literature that is related to the Bayesian designs and the sequential strategies. Bayesian techniques for life test planning are available in the literature. Chaloner and Verdinelli (1995) provided a comprehensive discussion and applications of Bayesian designs for both linear and nonlinear models. Zhang and Meeker (2005; 2006) considered Bayesian test planning and presented a general Bayesian planning framework, where the optimum plan minimizes the pre-posterior expectation of the posterior variance. Hong et al. (2015) provided two numerical approaches to evaluate the Bayesian criterion and solved the optimum planning problems. In their methodology, an optimum Bayesian design is determined by including the prior knowledge of unknown parameters.

A sequential design strategy can be adopted to conduct tests when test units are expensive or when the testing is time-consuming. From the non-Bayesian framework, Wu (1985) and Chaudhuri and Mykland (1995) proved the convergence of the sequentially computed ML estimators and the convergence of the sequential design in a nonlinear experiment. McLeish and Tosh (1990) proposed a sequential design via the ML estimation for probit models. From the aspect of Bayesian analysis in literature, several authors used the D-optimality as the desirable criterion, and then determined the optimum setting for the next design point.

Dror and Steinberg (2008) proposed an efficient procedure for a sequential experiment when the response is from a generalized linear model. Moreover, Hu (1998) and Roy et al. (2009) proved the convergence of the sequential design using the Bayesian estimates on nonlinear and binary response models, respectively. Zhu et al. (2014) proposed a sequential Bayesian strategy that converges to the locally D-optimality design corresponding to the true parameter values. For the purpose of a precise prediction, Azadi et al. (2014) provided an algorithm of sequential Bayesian designs on the application of surface electromyographic experiments for binary responses.

1.4 Overview

The rest of the paper is organized as follows. Section 2 introduces the physical model and develops the framework for the optimal design based on the statistical inference problem of interest. Section 3 addresses the criterion that is used to determine the optimum sequential Bayesian design. Section 4 shows the parameter estimation and the results of the sequential Bayesian design. Section 5 shows the comparison between the proposed and traditional optimum test planning methods and the sensitivity analysis on priors. Finally, Section 6 presents a summary of the results and possible areas for future work.

2 The General Framework for the Optimal Test Plan

2.1 Test Setup and Related Notations for Fatigue Testing

The most commonly-used method is the constant amplitude cyclic fatigue testing. In the test, the stress (σ) is applied to the sides of testing coupon, where the positive and negative values of applied stress represent tensile and compressive stresses, respectively. Three types of tests are typically used, which are tension-tension loading, compression-compression loading, and reverse loading tests. Figure 1 shows a plot of the stress over time under the three types of fatigue testing, where σ_M and σ_m denote the maximum and minimum stresses, respectively. The stress ratio is defined as $R = \sigma_m/\sigma_M$. The test is a tension-tension loading test if $0 < R < 1$, a compression-compression loading test if $1 < R < \infty$, and a reverse loading test if $-\infty < R < 0$. Examples are glass fibers of aircraft, helical compression springs, and railcar axles respectively. As illustrated in Figure 1, one cycle is defined as the smallest segment of the stress versus time which is repeated periodically. The testing unit is declared to have failed if it cracks or breaks after a period of cyclic loading. Because the fiber in the material has a direction, the angle between the testing direction and the testing coupon is a variable, depending on the fiber direction. We use α to denote the smallest angle between the testing direction and the fiber direction. Furthermore, the ultimate stress of a material is denoted by σ_{ult}, under which the material will break at its first cycle. Also, let h be the frequency of the cyclic stress testing, where the unit of the frequency is Hz.

In the fatigue testing literature, the maximum stress σ_M is used to represent the stress level of the test. That is, the design variable in our test planning is $x = \sigma_M$. The loading cycles are repeated until failure, and the cycles to failure t is recorded for each testing unit. In general, the number of cycles-to-failure decreases as the stress level increases. All other variables such as σ_{ult}, R, and h are fixed and set by the experimenter, and then σ_m is determined when R and x are specified. Also, the test will stop if the sample unit has not failed after a certain threshold (e.g., five millions of cycles), resulting in a right-censored observation.

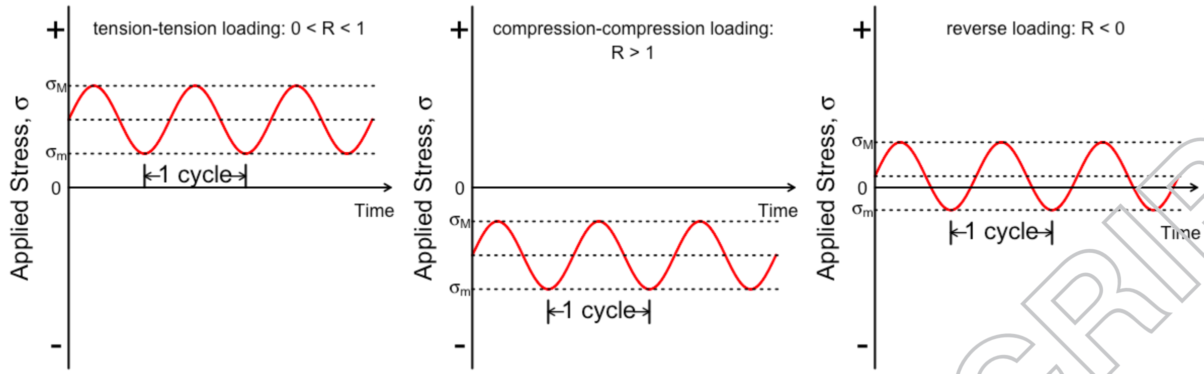

Figure 1: Illustrative diagram of constant amplitude testing.

2.2 Physical and Statistical Models

In the reliability literature (e.g., Pascual and Meeker, 1999), the cycles-to-failure random variable, T, is often described by a distribution of the log-location-scale family. The cumulative distribution function (cdf) and probability density function (pdf) of the distribution are given as

$$F(t;\boldsymbol{\theta}) = \Phi\left[\frac{\log(t)-\mu}{\nu}\right], \quad \text{and} \quad f(t;\boldsymbol{\theta}) = \frac{1}{\nu}\phi\left[\frac{\log(t)-\mu}{\nu}\right],$$

respectively. Here, $\Phi(\cdot)$ and $\phi(\cdot)$ are the cdf and pdf of the standard distribution (i.e., μ=0, ν=1), respectively. We use $\boldsymbol{\theta}$ to denote the unknown parameters in the distribution. The Weibull and lognormal distributions are common examples in the log-location-scale family. In fatigue modeling, the scale parameter ν is assumed to be constant and the location parameter $\mu = \mu_{\boldsymbol{\beta}}(x)$ is specified as a function of the stress x with parameters $\boldsymbol{\beta}$.

We use the physically motivated nonlinear model in Epaarachchi and Clausen (2003) to model $\mu = \mu_{\boldsymbol{\beta}}(x)$, which was derived from assumptions on the accumulation of fatigue damage in polymer composite materials. The fatigue model includes the effects of stress level, frequency, and the angle between the testing direction and the fiber direction, and the model is more suitable and flexible for various combination of experimental settings and composite materials. Then, the stress-life (typically referred to as "S-N" in fatigue literature) relationship is defined as

$$\mu_{\boldsymbol{\beta}}(x) = \frac{1}{B}\log\left\{\left(\frac{B}{A}\right)h^{B}\left(\frac{\sigma_{\text{ult}}}{x}-1\right)\left(\frac{\sigma_{\text{ult}}}{x}\right)^{\gamma(\alpha)-1}\left[1-\psi\left(R\right)\right]^{-\gamma(\alpha)}+1\right\}, \tag{1}$$

where $\mu_{\boldsymbol{\beta}}(x)$ is the cycles-to-failure at stress x and $\boldsymbol{\beta} = (A, B)'$. Note that A is related to environmental effects on the material fatigue and B is related to effects from the material itself. In addition, the function $\psi\left(R\right)$ is defined as $\psi(R) = R$ if $-\infty < R < 1$ and $\psi(R) = 1/R$ if $1 < R < \infty$, and $\gamma\left(\alpha\right) = 1.6 - \psi\left|\sin\left(\alpha\right)\right|$. Then, in the fatigue testing model, the unknown parameters are $\boldsymbol{\theta} = (\boldsymbol{\beta}', \nu)'$.

In reliability studies, a quantile in the lower tail of the failure-time distribution provides a meaningful life characteristic. In particular, let $\xi_{p,x}$ denote the pth quantile at the stress x, which is solved from $p = F(\xi_{p,x}; \boldsymbol{\theta})$. The log of the pth quantile can be expressed as

$$\log\left(\xi_{p,x}\right) = \mu_{\boldsymbol{\beta}}(x) + z_p\nu, \tag{2}$$

and z_p is the pth quantile of the standard log-location-scale distribution.

2.3 Objective of the Experiment and Basic Design Criterion

The objective of the experiment is to estimate the parameters $\boldsymbol{\theta}$ from the observed data, and substitute these estimators $\hat{\boldsymbol{\beta}}$ and $\hat{\nu}$ into (2) to obtain the estimator of $\log\left(\xi_{p,u}\right)$ at a normal stress level $x = u$, i.e.,

$$\log\left(\widehat{\xi}_{p,u}\right) = \mu_{\widehat{\boldsymbol{\beta}}}(u) + z_p\widehat{\nu}. \tag{3}$$

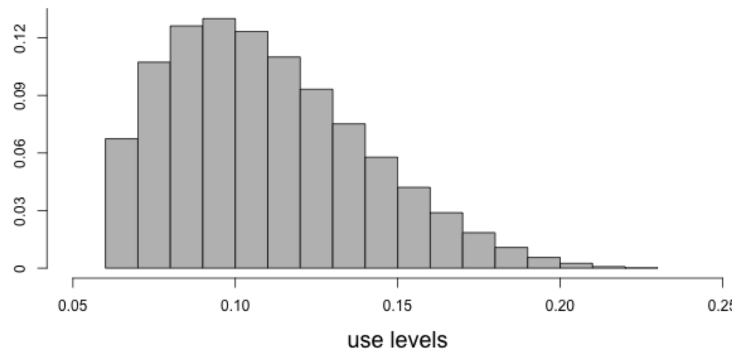

Figure 2: Plot of a distribution profile of use stress levels, ranging from 5% to 25% of the ultimate stress σ_{ult}.

However, in real-life applications, the normal stress level (to which polymeric materials are exposed) is not a constant. Rather, it is more likely that a material will experience various stress levels in use. Hence, we consider test planning under a use stress profile (i.e., multiple use conditions). Figure 2 gives an example of use stress profile. The use stress profile is represented by a set of specified use levels, $\{u_1, \cdots, u_K\}$, and their corresponding relative frequency $\{w_1, \cdots, w_K\}$, where $\sum_{k=1}^{K} w_k = 1$. Note that $(u_k, w_k), k = 1, \cdots, K$, are determined by the application and are fixed prior to choosing the optimal design. For a given use stress profile, the weighted sum of the asymptotic variance of the estimator of the pth quantile of the lifetime distribution at a vector of specified use levels can be expressed as

$$\sum_{k=1}^{K} w_k \ \text{AVar}\left[\log\left(\widehat{\xi}_{p,u_k}\right)\right].$$

For pre-specified $(u_k, w_k), k = 1, \cdots, K$, the above asymptotic variance depends on the input points $x_1, \ldots, x_n$, and will therefore be the basic component of the criterion or the utility function that will be used to determine $x_1, \ldots, x_n$. That is, the input points should be chosen so that the asymptotic variance is minimized. We discuss the computation of the asymptotic variance in the following section.

2.4 Maximum Likelihood Estimation and Asymptotic Variance

The data from a fatigue test are denoted by (x_i, t_i, δ_i), where x_i is the corresponding σ_M for unit i, t_i is the observed cycles to failure (or censored level), δ_i is the censoring indicator, and $i = 1, \ldots, n$. In particular, $\delta_i = 1$ if the unit is censored at t_i and $\delta_i = 0$ if the unit fails at t_i. A censored observation means the test sample had not failed at the end of the testing. Let $(\boldsymbol{x}_n, \boldsymbol{t}_n, \boldsymbol{\delta}_n)$ denote the observed data, where $\boldsymbol{x}_n = (x_1, \cdots, x_n)'$, $\boldsymbol{t}_n = (t_1, \cdots, t_n)'$, and $\boldsymbol{\delta}_n = (\delta_1, \cdots, \delta_n)'$. Then, the likelihood function is

$$L(\boldsymbol{\theta}|\boldsymbol{x}_n, \boldsymbol{t}_n, \boldsymbol{\delta}_n) = \prod_{i=1}^{n} \left\{ \frac{1}{\nu t_i} \phi \left[\frac{\log(t_i) - \mu_{\boldsymbol{\beta}}(x_i)}{\nu} \right] \right\}^{(1-\delta_i)} \left\{ 1 - \Phi \left[\frac{\log(t_i) - \mu_{\boldsymbol{\beta}}(x_i)}{\nu} \right] \right\}^{\delta_i},$$

and the log-likelihood function is

$$l(\boldsymbol{\theta}|\boldsymbol{x}_n, \boldsymbol{t}_n, \boldsymbol{\delta}_n) = \sum_{i=1}^{n} (1 - \delta_i) [\log \phi(z_i) - \log(\nu) - \log(t_i)] + \delta_i \log [1 - \Phi(z_i)], \qquad (4)$$

where $z_i = [\log(t_i) - \mu_{\boldsymbol{\beta}}(x_i)]/\nu$. The ML estimates $\widehat{\boldsymbol{\theta}}$ can be obtained by finding the values of $\boldsymbol{\theta}$ that maximize (4), and the Fisher information matrix is

$$I_n(\boldsymbol{\theta}) = I_n(\boldsymbol{\theta}, \boldsymbol{x}_n) = \mathrm{E}\left[-\frac{\partial^2 l(\boldsymbol{\theta})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'} \right].$$

The suffix n in $I_n(\boldsymbol{\theta}, \boldsymbol{x}_n)$ denotes that the information is based on n data points. The details of calculation and the resulting formulae for $I_n(\boldsymbol{\theta}, \boldsymbol{x}_n)$ are given in Supplementary Section 1. By the invariance property of ML estimators, the ML estimator of the pth quantile at stress level u can be obtained from (3) where $\widehat{\boldsymbol{\beta}}$ and $\widehat{\nu}$ are the ML estimators of $\boldsymbol{\beta}$ and ν, respectively. Given a specific stress level at use condition, u_k, the large-sample asymptotic variance of $\log(\widehat{\xi}_{p,u_k})$ is

$$\text{AVar}\left[\log\left(\widehat{\xi}_{p,u_k}\right)\right] = \boldsymbol{c}_k' \; \Sigma_{\boldsymbol{\theta}}(\boldsymbol{x}_n) \; \boldsymbol{c}_k, \tag{5}$$

where $\boldsymbol{c}_k = [\partial\mu_{\boldsymbol{\beta}}(u_k)/\partial A, \; \partial\mu_{\boldsymbol{\beta}}(u_k)/\partial B, \; z_p]'$ and $\Sigma_{\boldsymbol{\theta}}(\boldsymbol{x}_n) = I_n^{-1}(\boldsymbol{\theta}, \boldsymbol{x}_n)$.

Consequently, for a pre-specified use stress profile (u_k, w_k), $k = 1, \ldots, K$, the design criterion (weighted sum of the asymptotic variance of the estimator of the pth quantile of the lifetime distribution at a vector of specified use levels) can be expressed as

$$\sum_{k=1}^{K} w_k \; \text{AVar}\left[\log\left(\widehat{\xi}_{p,u_k}\right)\right] = \sum_{k=1}^{K} w_k \boldsymbol{c}_k' \; \Sigma_{\boldsymbol{\theta}}(\boldsymbol{x}_n) \; \boldsymbol{c}_k. \tag{6}$$

King et al. (2016) proposed a non-Bayesian optimum test planning technique based on (6) for polymer composites fatigue studies. Note that such optimal designs depend heavily on the parameter settings. However, the true parameter settings are usually unknown. In addition, the experiment is very costly. Therefore, we use (6) to develop our sequential Bayesian design, in which one design point is added at each step by optimizing the expectation of (6) over the posterior distribution of the parameters $\boldsymbol{\theta}$, given the data obtained till that step.

3 Test Planning Methodology

In the previous section, we have seen that the fundamental idea behind obtaining an optimal design, given a pre-specified use stress profile (u_k, w_k), $k = 1, \ldots, K$, is to optimize the basic design criterion given by (6) with respect to the input points x_i, $i = 1, \ldots, n$. For simplicity, we define the scaled design points for the test plan as $q_i = x_i/\sigma_{\text{ult}}$, which is the ratio of maximum stress to the ultimate tensile strength so that $0 < q_i \leq 1$. In practice, a planning range $[q_L, q_U]$ of the scaled design variable q is considered, where q_L and q_U are respectively the lower and upper bounds of q. As mentioned earlier, a sequential approach is most natural in polymer composite testing set-up. We need to specify prior distributions for the parameters, which we do next.

3.1 Prior Distribution

Prior distributions for the unknown parameters can be postulated on the basis of the information obtained from prior experiments, if available. As mentioned in Section 2, the parameter A represents the effect of environmental factors and the parameter B represents the material properties. Hence, both A and B have physical interpretations based on previous experiments, and their prior distributions are postulated to be $A \sim \text{Unif}(a_1, a_2)$ and $B \sim \text{Unif}(b_1, b_2)$, respectively, where a_1, a_2, b_1, and b_2 are known constants. In addition, for the prior distribution of ν^2, we postulate an inverse gamma distribution with shape parameter κ and scale parameter γ, where κ and γ are obtained from previous experience. This choice is motivated by the fact that the inverse gamma distribution is a conjugate prior distribution for ν^2 when the cycles-to-failure distribution is the lognormal distribution.

3.2 Criterion for Sequential Bayesian Designs

Inspired by this Bayesian c-optimality criterion (Chaloner and Verdinelli, 1995) and the recent work of Zhu et al. (2014), we propose the following objective function based on (6) for our sequential Bayesian design:

$$\varphi\left(q_{\text{new}}\right) = \int_{\boldsymbol{\Theta}} \left[\sum_{k=1}^{K} w_k \; \boldsymbol{c}_k' \; \Sigma_{\boldsymbol{\theta}}\left(\boldsymbol{q}_{\text{new}}\right) \boldsymbol{c}_k \right] \pi\left(\boldsymbol{\theta} | \boldsymbol{q}_n, \boldsymbol{t}_n, \boldsymbol{\delta}_n\right) d\boldsymbol{\theta}, \tag{7}$$

where $\Sigma_{\boldsymbol{\theta}}(\boldsymbol{q}_{\text{new}}) = [I_n(\boldsymbol{\theta}, \boldsymbol{q}_n) + I_1(\boldsymbol{\theta}, q_{\text{new}})]^{-1}$, $\boldsymbol{q}_{\text{new}} = (\boldsymbol{q}_n', q_{\text{new}})'$, and $\boldsymbol{q}_n = (q_1, \ldots, q_n)'$. Then, the optimum design for the $(n+1)$th design point is

$$q^*_{n+1} = \arg \min_{q_{\text{new}} \in [q_L, q_U]} \varphi(q_{\text{new}}). \tag{8}$$

Note that the posterior distribution under n observations is essentially treated as the prior information of parameters to determine the next design point.

From Theorem 1 of Hu (1998), we can show that our proposed method converges to the locally optimum design which is proposed in King et al. (2016). To satisfy the necessary condition of Theorem 1 of Hu (1998), it is easy to show that the mean of the cycles to failure distribution is bounded and continuous in an open set containing the design and the parameter spaces because the design and the parameter spaces are compact sets. In addition, the parameter space is often bounded from the prior information. Hence, by applying Theorem 1 and Remark 4 in Hu (1998), the proposed sequential Bayesian design converges to the optimum design based on the true values of parameters.

The implication of this discussion is that, even if one starts with planning information of parameters that is different from the truth, the estimator will converge to the true parameter values using the sequential Bayesian design strategy. Because the parameter will converge to the true values, the design also converges to the optimum design. Finally, we provide a discussion for proposing a Bayesian design criterion defined by taking the expectation of a frequentist criterion (asymptotic variance) over the prior distribution of parameters. As shown by Zhang and Meeker (2006), the approximated posterior variance is

$$C(D) = \int \mathbf{c}' \left[S^{-1} + I_D(\boldsymbol{\theta})\right]^{-1} \mathbf{c} \; d(\omega(\boldsymbol{\theta})), \tag{9}$$

where D is the specified design, $I_D(\boldsymbol{\theta})$ is the Fisher information matrix based on D, S

represents the covariance matrix of the prior distribution, and $\omega(\boldsymbol{\theta})$ is the prior distribution. The form of the proposed objective function (7) is similar to the form of (9), with the posterior distribution with n observations being essentially treated as the prior information prior to obtaining the next design point.

3.3 The Proposed Procedure

To optimize the Bayesian criterion (7), posterior samples need to be drawn from posterior distribution of $\boldsymbol{\theta}$ in the sequential updating process. Using the prior distributions mentioned in Section 3.1, the joint posterior distribution of (A, B, ν^2) is

$$\begin{aligned}
\pi\left(A, B, \nu^2 | \boldsymbol{q}_n, \boldsymbol{t}_n, \boldsymbol{\delta}_n\right) &\propto \prod_{i=1}^{n} \left\{ \frac{1}{\nu t_i} \phi\left[\frac{\log(t_i) - \mu_{\boldsymbol{\beta}}(x_i)}{\nu}\right] \right\}^{(1-\delta_i)} \left\{ 1 - \Phi\left[\frac{\log(t_i) - \mu_{\boldsymbol{\beta}}(x_i)}{\nu}\right] \right\}^{\delta_i} \\
&\quad \times \pi(A)\pi(B)\pi(\nu^2) \\
&= \prod_{i=1}^{n} \left\{ \frac{1}{\nu t_i} \phi\left[\frac{\log(t_i) - \mu_{\boldsymbol{\beta}}(x_i)}{\nu}\right] \right\}^{(1-\delta_i)} \left\{ 1 - \Phi\left[\frac{\log(t_i) - \mu_{\boldsymbol{\beta}}(x_i)}{\nu}\right] \right\}^{\delta_i} \\
&\quad \times \frac{\gamma^\kappa}{\Gamma(\kappa)} \left(\nu^2\right)^{-\kappa-1} e^{-\gamma/\nu^2} \mathbf{1}_{[a_1,a_2]}(A) \mathbf{1}_{[b_1,b_2]}(B).
\end{aligned}$$

Then, the conditional posterior distribution of ν^2 given (A, B) is

$$\begin{aligned}
\pi\left(\nu^2 | \boldsymbol{q}_n, \boldsymbol{t}_n, \boldsymbol{\delta}_n, A, B\right) &\propto \left(\frac{1}{\nu^2}\right)^{\frac{(n-\sum_{i=1}^{n}\delta_i)}{2}+\kappa+1} \exp\left\{ -\frac{\sum_{i=1}^{n}(1-\delta_i)\left[\log t_i - \mu_{\boldsymbol{\beta}}(x_i)\right]^2 + 2\gamma}{2\nu^2} \right\} \\
&\quad \times \left\{ 1 - \Phi\left[\frac{\log(t_i) - \mu_{\boldsymbol{\beta}}(x_i)}{\nu}\right] \right\}^{\delta_i},
\end{aligned}$$

and the conditional posterior distribution of (A, B) given ν^2 is

$$\pi\left(A, B|\boldsymbol{q}_n, \boldsymbol{t}_n, \boldsymbol{\delta}_n, \nu^2\right) \propto \prod_{i=1}^{n} \left\{\frac{1}{\nu t_i}\phi\left[\frac{\log\left(t_i\right) - \mu_{\boldsymbol{\beta}}(x_i)}{\nu}\right]\right\}^{(1-\delta_i)} \left\{1 - \Phi\left[\frac{\log\left(t_i\right) - \mu_{\boldsymbol{\beta}}(x_i)}{\nu}\right]\right\}^{\delta_i}$$
$$\times \mathbf{1}_{[a_1,a_2]}(A)\mathbf{1}_{[b_1,b_2]}(B).$$

Each iteration of the proposed sequential strategy involves three steps: (i) drawing samples from the posterior distribution, (ii) using a Monte Carlo method for approximating the criterion in (7), and (iii) optimizing the criterion. In this paper, we propose two algorithms to solve (i), (ii), and (iii) and these algorithms (**Algorithm 1** and **Algorithm 2**) are given in the supplementary material (Supplementary Sections 2 and 3).

Note that the proposed approach still works even if there is no current data but only the prior information. Specifically, the posterior distribution after n observations and information matrix in (7) are replaced by the prior distribution of parameters, $\pi(\boldsymbol{\theta})$, and the precision matrix of prior distribution (Hong et al., 2015), respectively. In some situations, instead of pre-fixing the required number of test samples, it may be desirable to define a stopping rule for the sequential design. One purpose of the sequential scheme is to increase the information on model parameters by obtaining data sequentially. Therefore, stopping rules can be defined in terms of the relative error of parameter estimates or the posterior distribution of parameters (Zhu et al., 2014) and the efficient estimates of parameters (Never, 1994). Once the difference of the criterion of the rules is smaller than a specified level, then the sequential scheme can be stopped.

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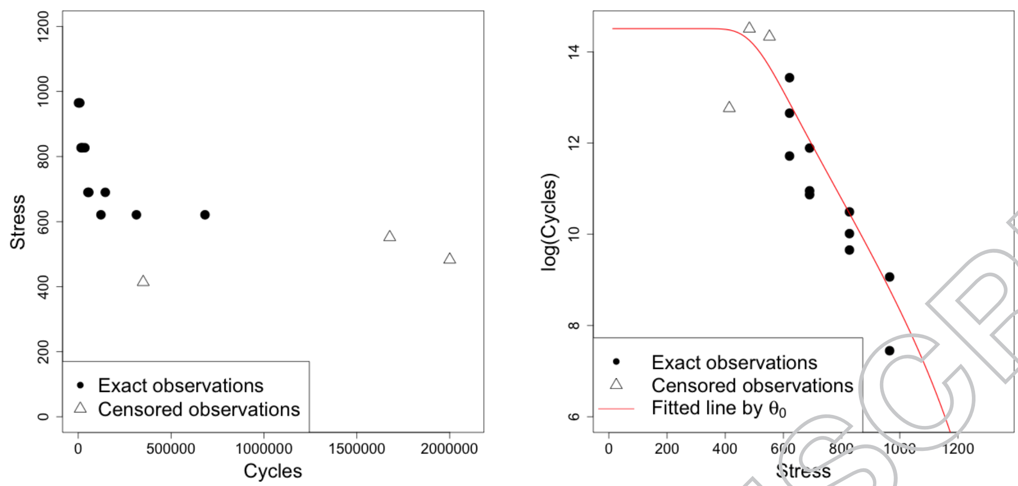


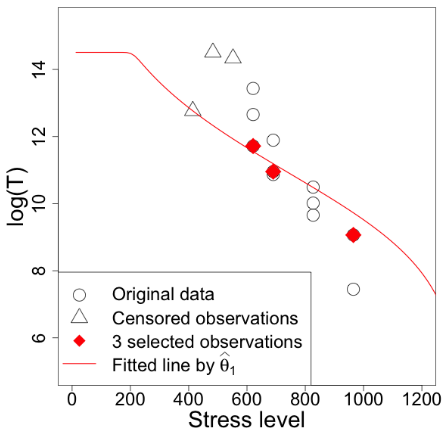
(a) Stress-life relationship (b) Log of cycles vs. Stress

Figure 3: Plots of historical data from a fatigue testing experiment for E-glass fibers.

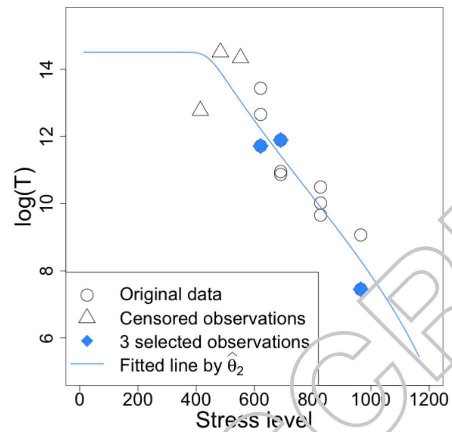
4 Application on Polymer composite tests Planning

In this section, we demonstrate the proposed sequential Bayesian algorithm in designing fatigue test experiments. Figure 3 shows 14 observations from a fatigue testing experiment for glass fibers (SNL/MSU/DOE Composite Materials Fatigue Database, 2016). Glass fiber is a composite material made of a polymer matrix reinforced with fibres. We consider a fatigue testing experiment for E-glass, the most common type of glass fiber, to demonstrate the proposed sequential Bayesian design procedure. The 14 observations that constitute the historical data include 3 right-censored and 11 failed observations. The other variables in (1) are set at $h = 2$, $R = 0.1$, $\alpha = 0$, and $\sigma_{\text{ult}} = 1339.67$ MPa. Figure 3(a) shows the stress-life relationship in original scale and Figure 3(b) shows the relationship between stress and the logarithm of cycles.

The ML estimates of $(A, B, \nu)'$ are $(0.0157, 0.3188, 0.7259)'$ and their asymptotic standard deviations are $(0.0056, 0.0434, 0.1456)'$, which are obtained by using the inverse of the Fisher information matrix. For the purpose of design evaluation and comparison using sim-



(a) Data subset 1



(b) Data subset 2

Figure 4: Plots show the three selected observations from the original data for data subset 1 (DS1) and data subset 2 (DS2). The observations of both subsets are selected under the same stress levels.

ulation studies, we will assume these values to be the true parameter values, denoted by $\boldsymbol{\theta}_0 = (A_0, B_0, \nu_0)' = (0.0157, 0.3188, 0.7259)'$ hereafter. We can also use Bayesian estimates based on the prior distributions. However such estimates are not useful in demonstrating how the proposed algorithm works. In polymer composite testing, the size of historical data is typically extremely limited. Thus we will demonstrate our proposed algorithm assuming that only three observations (a subset of the fourteen historical observations) are available to the experimenter prior to designing the experiment. Note that three is the smallest sample size that can be used to compute ML estimates of three parameters. The subset of three observations is shown in Figure 4(a) will be referred to as Data Subset 1 (DS1) henceforth. We also obtain a second subset of three observations shown in Figure 4(b) that will be referred to as Data Subset 2 (DS2), and be used later in Section 5.

Before implementing the sequential Bayesian algorithm, the following aspects of the design are finalized:

1. **Prior distributions of parameters**: Epaarachchi and Clausen (2003) provided the

estimates of A and B for different materials under different experimental settings, based on which we postulate the prior distributions as $A \sim \text{Unif}(10^{-6}, 0.1)$ and $B \sim \text{Unif}(10^{-6}, 1)$. From similar considerations, we assume ν^2 to follow an inverse gamma distribution with shape parameter $\kappa = 4.5$ and scale parameter $\gamma = 3$.

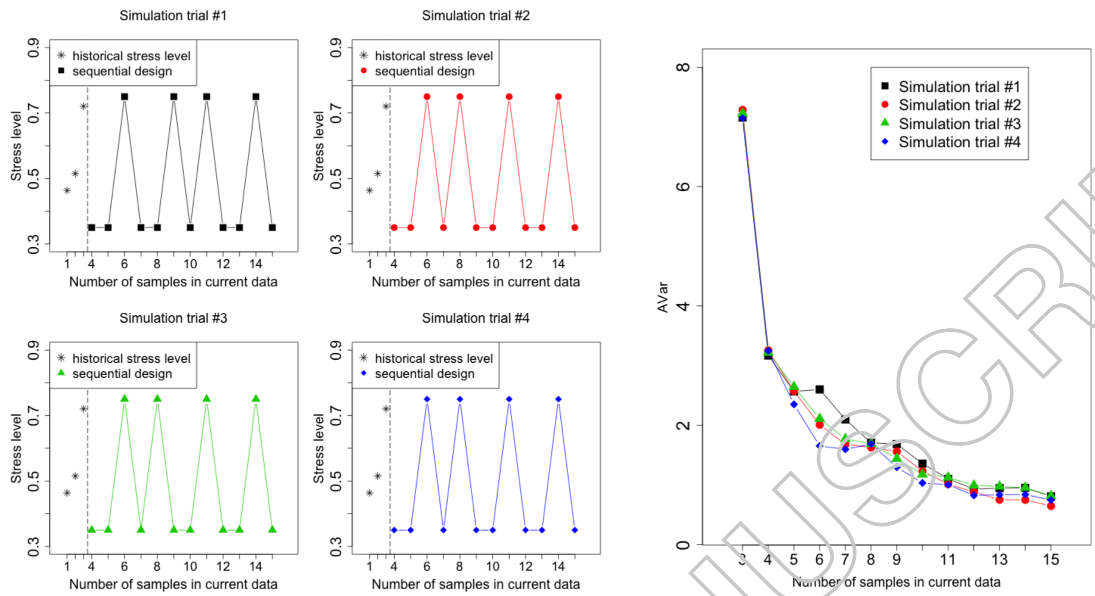
2. **Number of design points to be generated**: We consider a setting in which 12 new design points need to be determined sequentially given the historical data DS1.

3. **The candidate design points**: The candidate set $\{q^{(1)}, \ldots, q^{(L)}\}$ consists of points in the interval $[q_L = 0.35,\ q_U = 0.75]$ with a 5% increase.

4. **The use stress level profile**: The pattern of stress levels during potential use denoted by $(u_k, w_k), k = 1, \ldots, K$ is assumed to be the distribution shown in Figure 2, where $K = 20$, and the stress levels range from 5% to 25% of σ_{ult}.

We are now ready to apply our sequential Bayesian algorithm to obtain 12 new design points by applying **Algorithms 1** and **2**. The samples from the posterior distribution are drawn using the Markov chain Monte Carlo (MCMC) method described in **Algorithm 1**, and the illustrative auto-correlation functions (ACF) based on the original dataset are shown in Figure 4.1 in Supplementary Section 4. The weighted sum of the asymptotic variance in (6) is computed in **Algorithm 2**. In step 5 of **Algorithm 2**, the response t_{new} corresponding to a new design point q^*_{new} is obtained by generating a random draw from the cycles-to-failure distribution with the true values $\boldsymbol{\theta}_0$. Note that the cost of computation depends on the size of candidate set and the size of draws from the MCMC samples. The average computation time required to generate 12 new design points when 1,000 MCMC samples are drawn to compute the asymptotic variance for each candidate point.

The design points obtained sequentially and the corresponding values of the asymptotic variance are shown in Figure 5. Here, we illustrate the design properties using four specific



(a) Sequential Bayesian design trials based on DS1 (b) The values of asymptotic variance

Figure 5: Plots show sequential designs and corresponding values of asymptotic variance based on DS1.

simulation outputs. Figure 5(b) shows that the values of the asymptotic variance exhibit a monotonically decreasing trend after each new design point is added sequentially to the initial design DS1. From Figure 5(a), we also observe that the sequential Bayesian algorithm tends to converge to a design with support at two points (i.e., 0.35 and 0.75) with weights in the ratio of 2:1.

This observation can be explained by the fact that the design criterion demands a precise prediction at the use condition, and hence more design points are located at lower stress levels. We note that the locally optimum design derived by King et al. (2016) at $\boldsymbol{\theta} = \boldsymbol{\vartheta}_0$ would allocate exactly 8 and 4 points to the lower bound q_L and upper bound q_U, respectively. The results therefore suggest that our sequential algorithm converges to the two-level locally optimum design for polymer composites derived by King et al. (2016) corresponding to the assumed true values of parameters.

However, the true values of parameters will always be unknown in practice. A common way to derive locally optimal designs is to substitute the ML estimates of parameters from a previous experiment to obtain the optimal design. Because we assume the historical data to be DS1 in this example, these ML estimates are $\widehat{\boldsymbol{\theta}}_1 = (0.0005, 0.7429, 0.1658)'$. Then, the two-level locally optimum design based on $\widehat{\boldsymbol{\theta}}_1$ allocates 11 and 1 points to the stress levels 0.65 and 0.75, respectively, which is not even close to the result based on the true values of parameters. In other words, if one adopts the strategy of traditional optimum design, inaccurate planning values for model parameters may cause an extremely inefficient design and lead to unreliable inferences about fatigue life. From this point of view, the proposed sequential Bayesian design is a much more robust method compared to the locally optimum designs because it is seen to converge to the true optimal design.

5 Design Performance Evaluation

In this section, we further investigate the performance of the proposed method by comparing it with other state-of-the-art designs described in Section 1.2. These designs include two-level optimum designs, compromise designs, and equally-spaced designs. Here, we use these designs to compare performance with the proposed sequential Bayesian design. The two-level optimum and compromise designs are supposed to be determined by substituting values of the parameters that are assumed to be close to their true values. Here we will obtain these designs by substituting the ML estimates obtained from the preliminary dataset. The descriptions of these designs are given below.

1. Two-level optimum design (TOD): Given the values of parameters, the optimum design consists of support at two design points denoted by q_1 and q_2 with their corresponding sample size allocations n_1 and n_2, respectively. Letting $q_2 = q_U$, q_1 is a function of

the parameters and needs to be determined. Then, the decision variables are q_1, n_1 and n_2, and the optimum design is determined by minimizing the weighted asymptotic variance at use conditions.

2. Compromise design (CPD): Given the values of parameters, the compromise design consists of support at three design points denoted by q_1, q_m, and q_2 with their corresponding sample size allocations n_1, n_m, and n_2, respectively. Letting $q_m = (q_1+q_2)/2$, $q_2 = q_U$, and n_m prefixed, q_1 and q_m are functions of the parameters and need to be determined. Then, the decision variables are q_1, q_m, n_1 and n_2, and the optimum design is determined by minimizing the weighted asymptotic variance at use conditions.

3. Equally-spaced design (EQD): Four equally-spaced stress levels are chosen in the interval $[q_L, q_U]$.

In the following subsections, we compare the performance of the proposed sequential Bayesian design (SBD) with that of the three state-of-the-art designs via simulation study.

5.1 Simulation Procedure

In the simulation study, we use the data subsets in Figure 4 as the historical dataset to determine the sequential Bayesian, two-level optimum, compromise, and equally-spaced designs. Assume that the true parameter settings are $\boldsymbol{\theta}_0$. The total number of design points generated by each design is $N = 12$.

1. For each historical dataset (DS1 and DS2), twelve design points are generated sequentially by **Algorithm 2**. The two-level optimum and compromise designs are derived by substituting the ML estimates for the relevant dataset.

Table 1: ML estimates and standard deviations (in parentheses) for DS1 and DS2.

Parameter	$\widehat{A}$	$\widehat{B}$	$\widehat{\nu}$
$\widehat{\boldsymbol{\theta}}_1$	0.0005 (0.0004)	0.7429 (0.0907)	0.1658 (0.0677)
$\widehat{\boldsymbol{\theta}}_2$	0.0162 (0.0068)	0.3333 (0.0561)	0.4044 (0.1651)

2. For each design $\boldsymbol{q}_N = (q_1, \ldots, q_N)$, observations $(\boldsymbol{t}_N, \boldsymbol{\delta}_N)$ are generated, using the model with the assumed true values $\boldsymbol{\theta}_0$.

3. The parameters are estimated from the data $(\boldsymbol{q}_N, \boldsymbol{t}_N, \boldsymbol{\delta}_N)$ and the asymptotic variance is computed using the estimated values of parameters.

After conducting 1,000 simulations with each setting, the average of asymptotic variances are computed as the measure of performance for the four strategies.

5.2 Simulation Results

To determine the two-level optimum and compromise designs, we first obtain the ML estimates from DS1 and DS2, and denote them by $\widehat{\boldsymbol{\theta}}_1 = (0.0005, 0.7429, 0.1658)'$ and $\widehat{\boldsymbol{\theta}}_2 = (0.0162, 0.3333, 0.4044)'$, respectively. Table 1 summarizes the ML estimates for the DS1 and DS2 and their approximate standard deviations computed as square roots of the diagonal elements of the inverse of the Fisher information matrix. Note that $\widehat{\boldsymbol{\theta}}_2$ is closer to $\boldsymbol{\theta}_0$ than $\widehat{\boldsymbol{\theta}}_1$. In non-Bayesian approaches, it is expected that the optimum designs are different when substituted parameter values are different.

Given that 12 new design points need to be determined, the two-level optimum designs, compromise designs, and equally-spaced designs are shown in Table 2, where TOD_0, TOD_i and CPD_i are designs determined by $\boldsymbol{\theta}_0$ and $\widehat{\boldsymbol{\theta}}_i$. Under the true parameter values, assumed to be $\boldsymbol{\theta}_0$, the two-level optimum designs are 8 and 4 points allocated to stress levels 0.35 and 0.75, respectively. It is expected that the optimum design based on $\widehat{\boldsymbol{\theta}}_2$ is the same as the result based on the true values since $\widehat{\boldsymbol{\theta}}_2$ is close to $\boldsymbol{\theta}_0$. However, the resulting test plan has very different optimum stress levels and sample size allocations if we treat $\widehat{\boldsymbol{\theta}}_1$ as the true values of parameters. The optimum stress levels based on $\widehat{\boldsymbol{\theta}}_1$ are 11 and 1 samples allocated to stress levels 0.65 and 0.75, respectively.

On the other hand, we also obtain the sequential Bayesian design based on DS1 and DS2 using the proposed approach. To compare the designs generated by the sequential Bayesian algorithm, we summarize the average sample size allocations from 1,000 simulations and display the summary in Figure 6. It is found that the average sample size allocated to stress levels 0.35 and 0.75 are about 8 and 4, respectively. It is interesting to note that the results obtained from the proposed algorithm are almost the same as those obtained from the two-level optimum design based on the true parameters, and hence TOD_0 is the limiting design generated by the sequential Bayesian approach. Moreover, unlike the two-level optimum design, the results are insensitive to the historical datasets used to obtain initial estimates of the parameters.

On the other hand, the averages of asymptotic variance of SBD evaluated by the Bayes estimates are 0.7663 and 0.7170 for DS1 and DS2, respectively. However, the Bayesian inference includes the effect of prior distributions. Hence, for the fair comparison of non-Bayesian and Bayesian methods, we evaluate the asymptotic variance of SBD by ML estimates of the final generated data. Table 3 shows the average values of asymptotic variances for the competing designs and the proposed sequential Bayesian design over the 1,000 simulations. Based on DS1, the sequential Bayesian design is more advantageous than the two-level optimum and compromise designs based on $\widehat{\boldsymbol{\theta}}_1$ and the equally-spaced design. However, based on DS2, the proposed method is slightly worse than the two-level optimum design. Note

Table 2: Optimum design for TOD, CPD, and EQD.

Design	Parameter	Stress level	Sample size allocation
TOD		$(q_1^*,\ q_2)$	$(n_1^*,\ n_2^*)$
TOD_0	$\boldsymbol{\theta}_0$	(0.35, 0.75)	(8, 4)
TOD_1	$\widehat{\boldsymbol{\theta}}_1$	(0.65, 0.75)	(11, 1)
TOD_2	$\widehat{\boldsymbol{\theta}}_2$	(0.35, 0.75)	(8, 4)
CPD		$(q_1^*,\ q_m^*,\ q_2)$	$(n_1^*,\ n_m,\ n_2^*)$
CPD_1	$\widehat{\boldsymbol{\theta}}_1$	(0.65, 0.70, 0.75)	(10, 1, 1)
CPD_2	$\widehat{\boldsymbol{\theta}}_2$	(0.35, 0.55, 0.75)	(7, 1, 4)
EQD		$(q_L,\ q_2,\ q_3,\ q_H)$	$(n_L,\ n_2,\ n_3,\ n_H)$
EQD		(0.35, 0.50, 0.60, 0.75)	(3, 3, 3, 3)

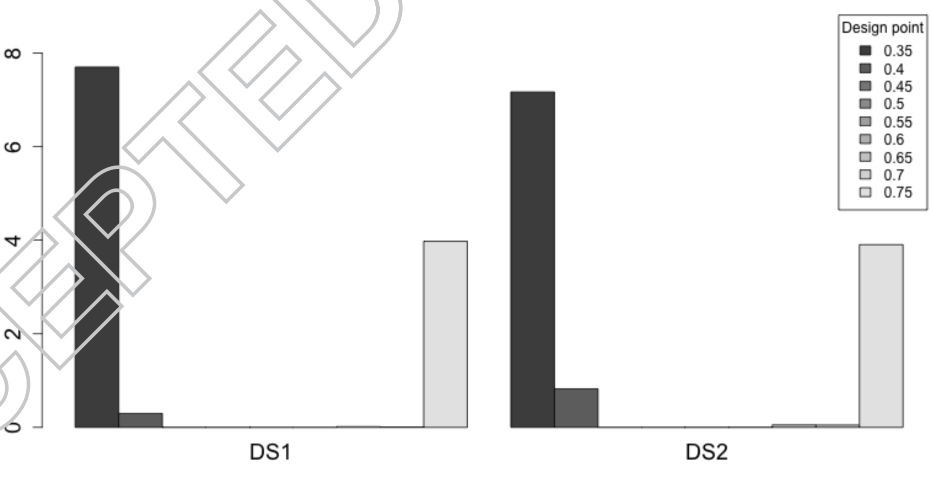


Figure 6: Plots show average sizes allocation of sequential Bayesian designs for the DS1 and DS2.

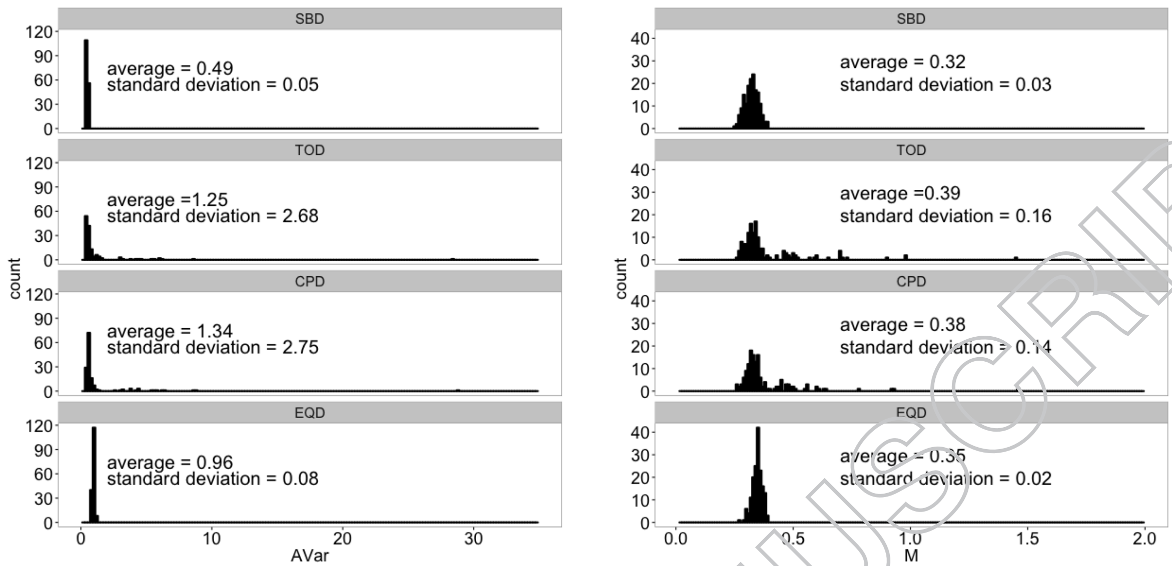
Table 3: The average of AVars for each design based on DS1 and DS2.

Design	DS1	Design	DS2
TOD_0	0.6236	TOD_0	0.4276
SBD	0.6163	SBD	0.4366
TOD_1	4.0337	TOD_2	0.4240
CPD_1	3.9219	CPD_2	0.4916
EQD	1.0077	EQD	0.8078

that the asymptotic variance of TOD_0 is not the smallest because the asymptotic variance is evaluated by the estimates of the data including the three historical observations and the new generated data. Hence, the proposed sequential Bayesian design performs well when true parameters are unknown.

5.3 Comparison Based on Different Historical Data

In previous sections, we found that the optimum designs based on non-Bayesian approaches depend strongly on the chosen values of the parameters. For a comprehensive comparison, we take all combinations of three failed observations from the original dataset in Figure 3 as the historical data, and determine the proposed sequential Bayesian design and the three competing designs based on their ML estimates of parameters for 300 simulation trials. After determining the optimum designs, we follow the simulation procedure described in Section 5.1. The average and standard deviation of asymptotic variance for all historical datasets are summarized in Figure 7(a). Among all historical datasets, the average values of asymptotic variance are tightly distributed around 0.49 and 0.96 for the sequential Bayesian design and equally-spaced design, whereas they vary widely for the two-level optimum and compromise designs. It is clear that the average and standard deviation of asymptotic variance for the sequential Bayesian design are both smaller than those for the three competing designs.



(a) Histograms of AVar (b) Histograms of M

Figure 7: Plots show comparisons of values of asymptotic variance and M of SBD, TOD, CPD, and EQD for all historical datasets.

As additional performance measures, two measures of relative error are defined to compare the estimated values of parameters after collecting 12 new generated observations. These are

$$m(\theta_j) = \frac{1}{S}\sum_{s=1}^{S}\left(\frac{|\hat{\theta}_{j,s} - \theta_j|}{\theta_j}\right)^2, \quad \text{and} \quad M = \sqrt{\sum_{j=1}^{3} m(\theta_j)},$$

where $\hat{\theta}_{j,s}$ is the estimated value of θ_j in the sth simulation trial, $\theta_j \in \boldsymbol{\theta}$, $j = 1, 2, 3$, $s = 1, \ldots, S$, and $S = 300$. Figure 7(b) shows the performance of the estimated parameters for all combinations of historical data. The average and the standard deviation of M for the SBD are smaller than those for the TOD and CPD but fairly close to those for EQD. It is clear that the relative errors of estimated values of $\boldsymbol{\theta}$ by the SBD are much smaller than those by the TOD and CPD.

From Figure 8(a), we further observe that the estimated values of A and B obtained by using the TOD and CPD depart substantially from the true values, while the estimated values of A and B obtained by using the SBD are close to the true values for all combinations of historical data. Figure 8(b) shows that the asymptotic variance is large when using the TOD and CPD due to inaccurate planning values. In contrast, the asymptotic variance and estimated values of parameters perform more reasonably by using the EQD and SBD.

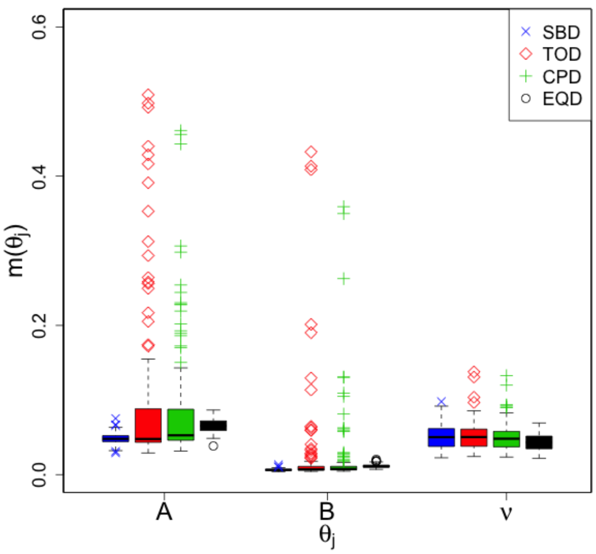
Besides, the histograms of average sample size allocation generated by SBD and TOD via simulation are shown in Figure 5.1 of Supplementary Section 5. For the sequential Bayesian designs, there are about 4 design points allocated at the highest stress level and about 8 design points allocated at lower stress levels which is mainly at stress level 0.35. As mentioned in the previous section, the optimum allocation and stress levels after conducting sequential Bayesian designs are almost the same as the two-level optimum design under the true parameters. When the substituted parameter values differ from the true values of the parameters, the TOD performs poorly. Generally speaking, the proposed sequential Bayesian design is a robust method especially for historical datasets with small sample sizes.

5.4 Sensitivity Analysis on Different Priors

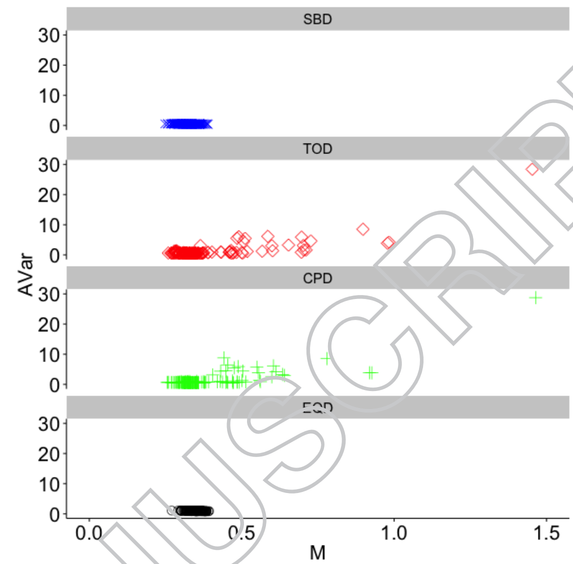
In this section, we compare the performance of the proposed method by selecting prior distributions that are different from the uniform priors postulated in Section 4. The following four prior distributions of A and B are considered.

- Prior 1: $A \sim \mathrm{N}(0.08, 0.0008)$ and $B \sim \mathrm{N}(1, 0.0833)$
- Prior 2: $A \sim \mathrm{N}(0.08, 0.0008)$ and $B \sim \mathrm{N}(0.3188, 0.0833)$
- Prior 3: $A \sim \mathrm{N}(0.0157, 0.0008)$ and $B \sim \mathrm{N}(1, 0.0833)$
- Prior 4: $A \sim \mathrm{N}(1, 0.25)$ and $B \sim \mathrm{N}(1, 0.25)$

The four prior distributions include at least one distribution that the true parameter is in the extreme tail of the prior as shown in Figure 6.1 in Supplementary Section 6. Note that the variances of the first three prior distributions are the same uniform priors as $A \sim$



(a) Boxplots of $m(\theta_j)$ for SBD and TOD.



(b) Scatter plots of M against AVar

Figure 8: Plots show comparisons of $m(\theta_j)$ and estimated values against asymptotic variance of SBD and TOD for all historical datasets.

Unif$(10^{-6}, 0.1)$ and $B \sim$ Unif$(10^{-6}, 1)$ postulated in Section 4. We use the same simulation procedure described in Section 4 to derive the sequential Bayesian design using the current data, DS1, using the four prior distributions. The results are shown in Table 4 and Figure 9. As seen from Figure 9, all the four priors lead to an approximate average allocation of 4 and 8 design points to 0.75 and 0.35, respectively. In addition, Table 4 shows that while the averages of asymptotic variances (AVar) obtained by using the ML estimates are the same (around 0.61) for all prior distributions, those obtained by using the Bayesian estimates are different depending on different priors. These findings are consistent with the observation of Pronzato and Pázman (2013), who pointed out the less sensitivity on the choice of priors at the design stage in the sequential Bayesian design.

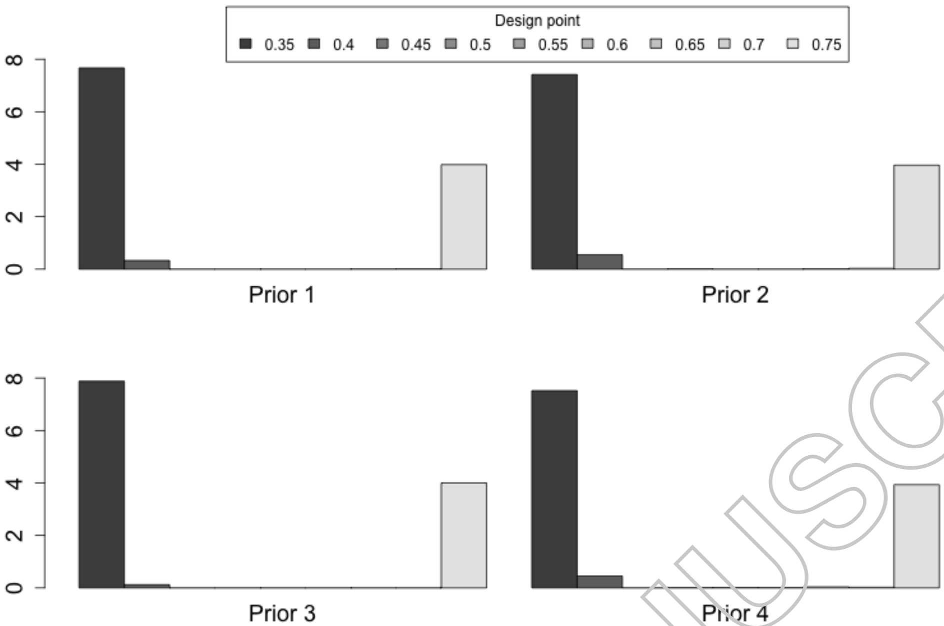


Figure 9: Plots show average sizes allocation of sequential Bayesian designs for different priors.

Table 4: The average of AVars for each prior distribution based on DS1.

Prior	AVar by Bayes estimates	AVar by ML estimates
Prior 1	0.7826	0.6048
Prior 2	0.8250	0.6162
Prior 3	0.7511	0.6076
Prior 4	0.7908	0.6068

6 Conclusion and Areas for Future Research

In this paper, we combine Bayesian techniques with a sequential test planning strategy to design fatigue tests. The design criterion based on the asymptotic variance of a predicted quantile of interest is used. A numerical algorithm is provided to determine the optimum stress level sequentially for the subsequent design point. From the numerical results and simulations, the proposed sequential Bayesian design is found to be superior to the existing strategies because of its robustness with respect to the historical data used to obtain preliminary parameter estimates. Furthermore, the results also show that the optimum stress levels and sample size allocation by sequential Bayesian designs quickly converge to the traditional two-level optimum design under the true parameters. The developed methodology is implemented in an R package "SeqBayesDesign" (Lee and Hong, 2018). The package can be applied to determine the sequential Bayesian design for traditional ALTs and the constant amplitude fatigue test based on the log-normal distribution and the Weibull distribution. Practitioners can implement it by inputting the original data settings and the prior information.

One may be concerned about the time it takes to complete the sequential design. In the illustrative example, the frequency f is 2 cycles per second. The exact experimental time of a testing unit is about 12 days (2,000,000/2 = 1,000,000 seconds) if the unit is censored. For one simulation trial with 12 specimens, the total experimental time is 11,470,956 cycles including three censored testing units. Consequently, it takes about 67 days to complete the sequential design. While occasionally there might be an urgency to obtain the test data, recall that our work is primarily motivated to cater to the requirements of industrial laboratories that have a limited number of testing machines and are compelled to test one sample at a time. In such situations, sequential designs are inevitable.

Although our methodology is motivated by polymer composite testing problems, the developed strategy can be easily extended to other ALT problems. For example, the proposed methodology can be applied to cases where doing much simultaneous testing is not practical, because the testing involves expensive equipment, and companies are often limited by the number of test machines that they can use simultaneously. In our methodology, we pick one design point in each updating step. In some situations, a company may have two or more testing machines available, our methods can be easily extended to batch sequential plans that allow optimal selection of two or more design points at each iteration.

The following are possible areas for future works:

1. One can consider sequential design based on the dual objective optimization. At the beginning of an experiment, the sample size is limited, often resulting in imprecise parameter estimation. Once enough observations are available, the main concern is to make precise predictions. Such a goal can be achieved by combining two different design criteria such as D- and c- optimality, as in Pan and Yang (2014).

2. One may consider a pre-posterior expectation of the criterion function or an expected utility function as the criterion in the sequential Bayesian design. Instead of using approximated formulation of the criterion, the efficient approaches and algorithms recently proposed by Weaver et al. (2016) and Overstall and Woods (2017) can be applied to the stages of evaluating the criterion and determining the next design point.

3. Pascual and Meeker (1998) considered modified sudden death test plans to address the problem of limited testing machine in life tests. Application of the proposed sequential updating method to such settings may be of interest.

4. Beside ALTs, the technique of accelerated degradation tests, including accelerated repeated measures degradation tests and accelerated destructive degradation tests, have also been widely used in the field. The planning of these types of accelerated degradation tests have been addressed in literatures (Shi et al., 2009; Shi and Meeker, 2012, 2013; Weaver and Meeker, 2014). In the future, the proposed method can be applied to the planning of such accelerated degradation tests.

Supplementary Material

The supplementary material including additional details: formulation of Fisher information matrix, proposed algorithms, and figures is available online (pdf file).

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